Majority-vote on undirected Barabási-Albert networks

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Abstract: On Barabási-Albert networks with z neighbours selected by each added site, the Ising model was seen to show a spontaneous magnetisation. This spontaneous magnetisation was found below a critical temperature which increases logarithmically with system size. On these networks the majority-vote model with noise is now studied through Monte Carlo simulations. However, in this model, the order-disorder phase transition of the order parameter is well defined in this system and this wasn't found to increase logarithmically with system size. We calculate the value of the critical noise parameter q_c for several values of connectivity z of the undirected Barabási-Albert network. The critical exponentes β/ν, γ/ν and $1/\nu$ were calculated for several values of z.

Keywords: Monte Carlo simulation, vote , networks, nonequilibrium.

Introduction

It has been argued that nonequilibrium stochastic spin systems on regular square lattice with up-down symmetry fall in the universality class of the equilibrium Ising model [1]. This conjecture was found in several models that do not obey detailed balance [2, 3, 4]. Campos *et al.* [5] investigated the majority-vote model on small-world network by rewiring the two-dimensional square lattice. These small-world networks, aside from presenting quenched disorder, also possess long-range interactions. They found that the critical exponents γ/ν and β/ν are different from the Ising model and depend on the rewiring probability. However, it was not evident whether the exponent change was due to the disordered nature of the network or due to the presence of long-range interactions. Lima *et al.* [6] studied the majority-vote model on Voronoi-Delaunay random lattices with periodic boundary conditions. These lattices posses natural quenched disorder in their conecctions. They showed that presence of quenched connectivity disorder is enough to alter the exponents β/ν and γ/ν the pure model and therefore that is a relevant term to such non-equilibrium phase-transition. Sumour and Shabat [7, 8] investigated Ising models on directed Barabási-Albert networks [9] with

the usual Glauber dynamics. No spontaneous magnetisation was found, in contrast to the case of undirected Barabási-Albert networks [10, 11, 12] where a spontaneous magnetisation was found lower a critical temperature which increases logarithmically with system size. Lima and Stauffer [13] simulated directed square, cubic and hypercubic lattices in two to five dimensions with heat bath dynamics in order to separate the network effects form the effects of directedness. They also compared different spin flip algorithms, including cluster flips [15], for Ising-Barabási-Albert networks. They found a freezing-in of the magnetisation similar to [7, 8], following an Arrhenius law at least in low dimensions. This lack of a spontaneous magnetisation (in the usual sense) is consistent with the fact that if on a directed lattice a spin S_j influences spin S_i, then spin S_i in turn does not influence S_j, and there may be no well-defined total energy. Thus, they show that for the same scale-free networks, different algorithms give different results. More recently, Lima [14] investigated the majority-vote model on directed Barabási-Albert network and calculated the β/ν, γ/ν, and $1/\nu$ exponents and these was different from the Ising model and depend on the values of connectivity z of the directed Barabási-Albert network. Now, we calculate the same β/ν, γ/ν, and $1/\nu$ exponents for majority-vote model on *undirected* Barabási-Albert network and these are different from the Ising model and depend on the values of connectivity z of the directed Barabási-Albert network.

Model and Simulaton

We consider the majority-vote model, on directed Barabási-Albert Networks, defined [16, 17, 6, 18] by a set of "voters" or spins variables σ taking the values $+1$ or -1, situated on every site of an undirected Barabási-Albert Networks with N sites, and evolving in time by single spin-flip like dynamics with a probability w_i given by

$$w_i(\sigma) = \frac{1}{2}\Big[1 - (1-2q)\sigma_i S\Big(\sum_{\delta=1}^{k_i} \sigma_{i+\delta}\Big)\Big], \tag{1}$$

where $S(x)$ is the sign ± 1 of x if $x \neq 0$, $S(x) = 0$ if $x = 0$, and the sum runs over all nearest neighbours of σ_i. In this network, each new site added to the network selecting z, already existing sites as neighbours influencing it; the

newly added spin does not influence these neighbours. The control parameter q plays the role of the temperature in equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbours.

To study the critical behavior of the model we define the variable $m = \sum_{i=1}^{N} \sigma_i / N$. In particular , we were interested in the magnetisation, susceptibility and the reduced fourth-order cumulant:

$$M(q) = [< |m| >]_{av}, \tag{2}$$

$$\chi(q) = N[< m^2 > - < |m| >^2]_{av}, \tag{3}$$

$$U(q) = \left[1 - \frac{< m^4 >}{3 < |m| >^2} \right]_{av}, \tag{4}$$

where $< ... >$ stands for a thermodynamics average and $[...]_{av}$ square brackets for a averages over the 20 realizations.

These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$M = N^{-\beta/\nu} f_m(x)[1 + ...], \tag{5}$$

$$\chi = N^{\gamma/\nu} f_\chi(x)[1 + ...], \tag{6}$$

$$\frac{dU}{dq} = N^{1/\nu} f_U(x)[1 + ...], \tag{7}$$

where ν, β, and γ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x = (q - q_c) N^{1/\nu} \tag{8}$$

being the scaling variable, and the brackets $[1 + ...]$ indicate corretions-to-scaling terms. Therefore, from the size dependence of M and χ we obtained the exponents β/ν and γ/ν, respectively. The maximum value of susceptibility also scales as $N^{\gamma/\nu}$. Moreover, the value of q for which χ has a maximum, $q_c^{\chi_{max}} = q_c(N)$, is expected to scale with the system size as

$$q_c(N) = q_c + bN^{-1/\nu}, \tag{9}$$

were the constant b is close to unity. Therefore, the relations (7) and (9) are used to determine the exponente $1/\nu$. We have checked also if the calculated exponents satisfy the hyperscaling hypothesis

$$2\beta/\nu + \gamma/\nu = D_{eff} \tag{10}$$

in order to get the effective dimensionality, D_{eff}, for various values of z.

We have performed Monte Carlo simulation on *undirected* Barabási-Albert networks with various values of connectivity z. For a given z, we used systems of size $N = 1000$, 2000, 4000, 8000, and 16000. We waited 10000 Monte Carlo steps (MCS) to make the system reach the steady state, and the time averages were estimated from the next 10000 MCS. In our simulations, one MCS is accomplished after all the N spins are updated. For all sets of parameters, we have generated 20 distinct networks, and have simulated 20 independent runs for each distinct network.

Results and Discussion

In Fig. 1 we show the dependence of the magnetisation M and the susceptiblity χ on the noise parameter, obtained from simulations on *undirected* Barabási-Albert network with 16000 sites and several values of connectivity z. In the part (a) each curve for M, for a given value of N and z, suggests that there is a phase transition from an ordered state to a disordered state. The phase transition occurs at a value of the critical noise parameter q_c, which is an increasing function the connectivity z of the directed Barabási-Albert network. In the part (b) we show the corresponding behavior of the susceptibility χ, the value of q where χ has a maximum is here identified as q_c. In Fig. 2 we plot Binder's fourth-order cumulant for different values of N and two different values of z. The critical noise parameter q_c, for a given value of z, is estimated as the point where the curves for different system sizes N intercept each other. In Fig 3 the phase diagram is shown as a function of the critical noise parameter q_c on connectivity z obtained from the data of Fig. 2.

The phase diagram of the majority-vote model on *undirected* Barabási-Albert network shows that for a given network (fixed z) the system becomes ordered for $q < q_c$, whereas it has zero magnetisation for $q \geq q_c$. We notice that the increase of q_c in function the z is slower of the one than in [18]. In the Fig. 4 and 5 we plot the dependence of the magnetisation and susceptibility, respectively at $q = q_c$ with the system size. The slopes of curves correspond to the exponent ratio β/ν and γ/ν of according to Eq. (5) and (6), respectively. The results show that the exponent ratio β/ν increase and γ/ν decrease at q_c when z increase, see Table I.

In Fig. 6 we display the scalings for susceptibility at $q = q_c(N)$ (square), $\chi(q_c(N))$, and for its maximum amplitude, χ_N^{max}, and the scalings for susceptibility at the $q = q_c$ obtained from Binder's cumulant, $\chi(q_c)$ (circle), versus

N for connectivity $z = 8$. The exponents ratio γ/ν are obtained from the slopes of the straight lines. For almost all the values of z, the exponents γ/ν of the two estimates agree (along with errors), see Table I. An increased z means a tendency to decrease the exponent ratio γ/ν, see Table I, they agree with the results of Luiz et al [18], but disagree with the results of Lima [14] for *directed* Barabási-Albert network, where the values of the exponents ratio γ/ν are all different and with a slight tendency to increase at q_c and decrease at $q_c(N)$. Therefore we can use the Eq. (9) , for fixed z, obtain the critical exponent $1/\nu$, see Fig. 7. In the Fig. 8 we show the critical behavior of β/ν, γ/ν and $1/\nu$ the exponentes in function of connectivity z.

To obtain the critical exponent $1/\nu$, we calculated numerically $U^{'}(q) = dU(q)/dq$ at the critical point for each values of N at connectivity fixed z. The results are well in agreement with the scaling relation (7) . Then , we also can calculate the exponents $1/\nu$, through this relation. Therefore we do not have to get the values of the exponents $1/\nu$ for each connectivity z

The Table I summarizes the values of q_c, the exponents β/ν, γ/ν, and the effective dimensionality of systems. For all values of z, $D_{eff} = 1$, which has been obtained from the Eq. (9), therefore when z increases, β/ν increases and γ/ν decreases at q_c, thus providing the value of $D_{eff} = 1$ (along with errors). Therefore, the *undirected* Barabási-Albert network has the same effective dimensionality that Erdös-Rényi's random graphs [18] and *directed* Barabási-Albert network [14]. J. M. Oliveira [16] showed which majority-vote model defined on regular lattice has critical exponents that fall into the same class of universality as the corresponding equilibrium Ising model. Campos et al [5] investigated the critical behavior of the majority-vote on small-world networks by rewiring the two-dimensional square lattice, Luiz et al [18] studied this model on Erdös-Rényi's random graphs, and Lima et al [6] also studied this model on Voronoy-Delaunay lattice and Lima *directed* Barabási-Albert network [14]. The results obtained these authors show that the critical exponents of majority-vote model belong to different universality classes.

Finally, we remark that our MC results obtained on *undirected* Barabási-Albert network for majority-vote model show that critical exponents are different from the results of [16] for regular lattice, of Luiz et al [18] for Erdös-Rényi's random graphs and Lima [14].

Conclusion

In conclusion, we have presented a very simple nonequilibrium model on

z	q_c	β/ν	γ/ν^{q_c}	$\gamma/\nu^{q_c(N)}$	$1/\nu$	D_{eff}
2	0.167(3)	0.036(8)	0.828(6)	0.805(11)	0.76(3)	0.90(1)
3	0.259(2)	0.133(21)	0.713(18)	0.655(31)	0.83(7)	0.979(27)
4	0.306(3)	0.231(22)	0.537(8)	0.519(17)	0.43(2)	0.999(23)
6	0.355(2)	0.283(8)	0.445(15)	0.423(3)	0.35(5)	1.011(17)
8	0.380(6)	0.323(2)	0.358(7)	0.405(6)	0.39(3)	1.004(7)
10	0.396(3)	0.338(2)	0.324(2)	0.380(3)	0.324(5)	1.000(2)
20	0.428(2)	0.344(2)	0.305(2)	0.350(2)	0.307(5)	0.993(2)
50	0.456(3)	0.366(2)	0.255(2)	0.341(3)	0.30(1)	0.987(2)
100	0.471(3)	0.373(2)	0.218(5)	0.330(3)	0.308(4)	0.964(5)

Table 1: The critical noise q_c, the critical exponents, and the effective dimensionality D_{eff} , for undirected Barabási-Albert network with connectivity z. Error bars are statistical only.

undirected Barabási-Albert network [7, 8]. Different of of Ising model, in these networks, the majority-vote model presents a second-order phase transition which occurs with connectivity $z > 1$. The exponents obtained are different from the other models. Nevertheless, our Monte Carlo simulations have demonstrated that the effective dimensionality D_{eff} equals unity, for all values of z, which agrees with the results de Luiz et al [18]. However, when z grows, the exponents in the critical point q_c, β/ν obtained through Binder's cumulant grows and exponents γ/ν decrease, satisfying the hyperscaling relation with $D_{eff} = 1$.

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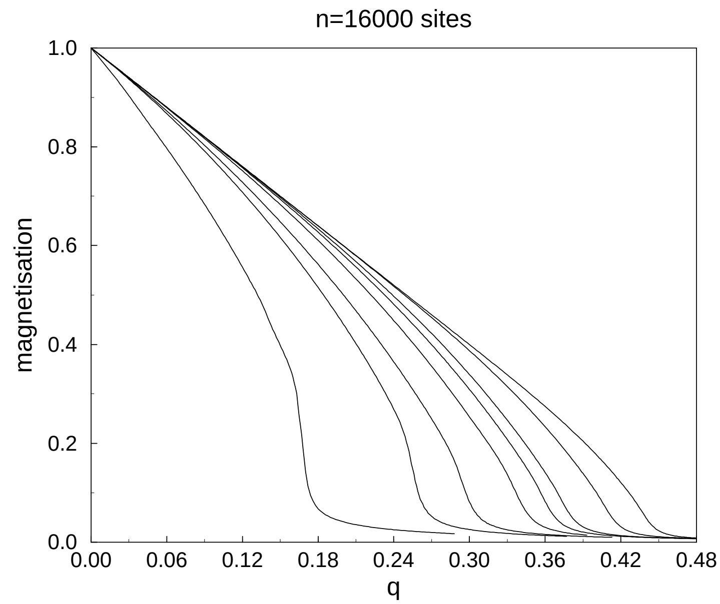

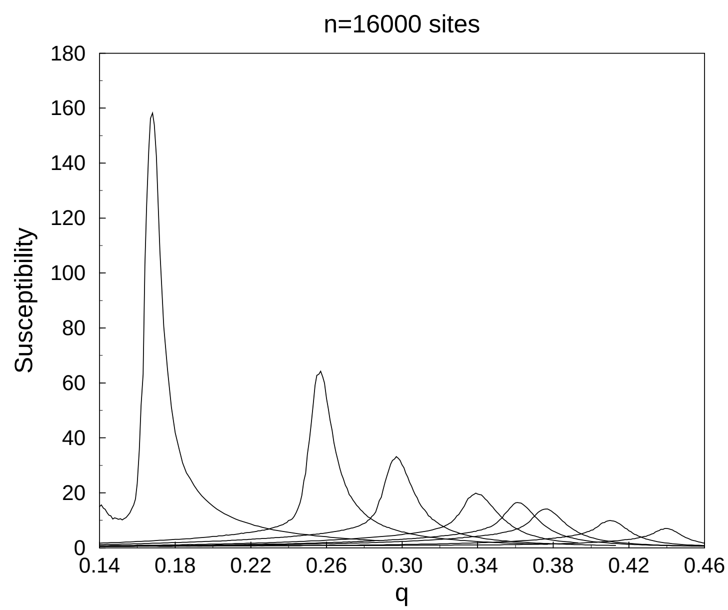

Figure 1: Magnetisation and susceptibility as a function of the noise parameter q, for $N = 16000$ sites. From left to right we have $z = 2$, 3, 4, 6, 8, 10, 20, and 50 .

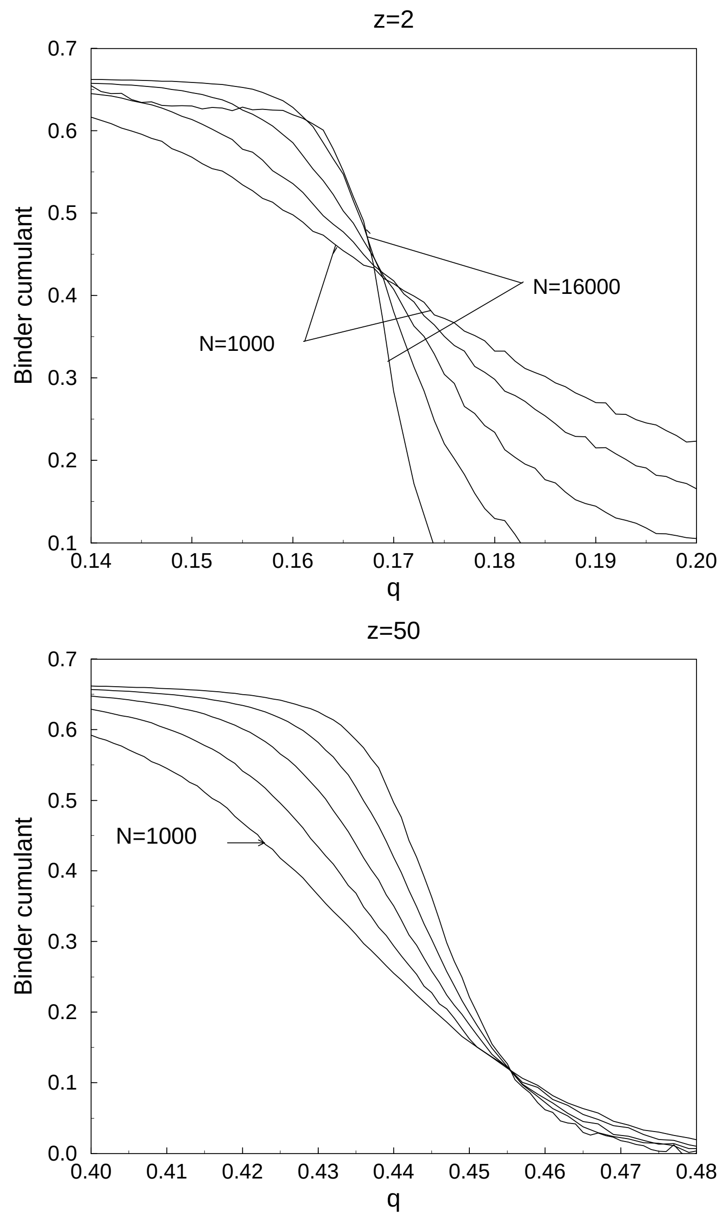

Figure 2: Binder's fourth-order cumulant as a function of q. We have $z = 2$ and $z = 50$ for $N = 1000$, 2000, 4000, 8000 and 16000 sites.

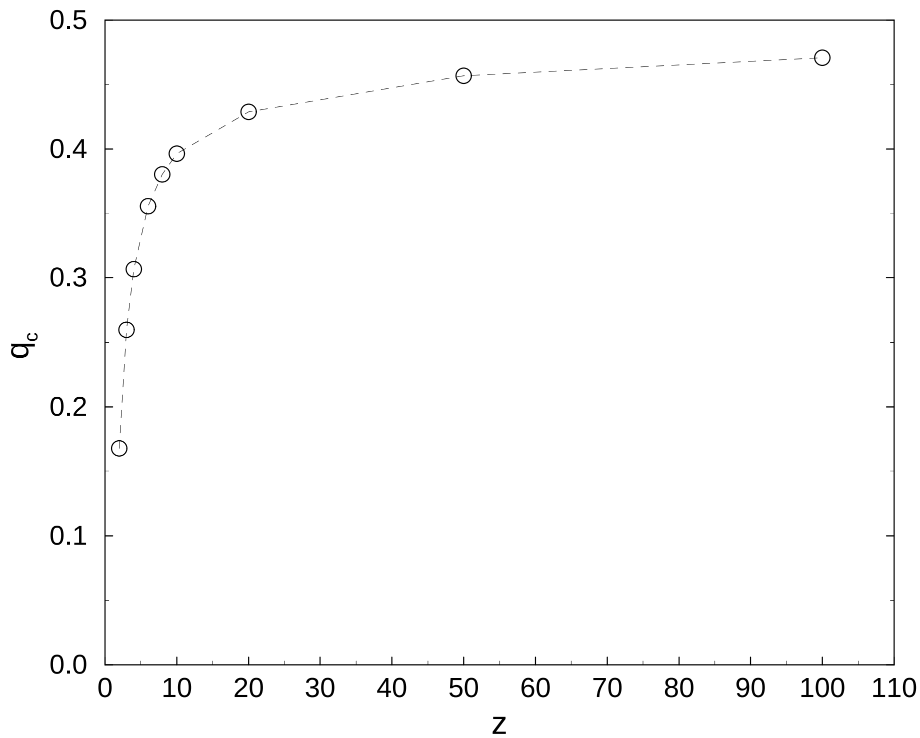

Figure 3: The phase diagram, showing the dependence of critical noise parameter q_c on connectivity z.

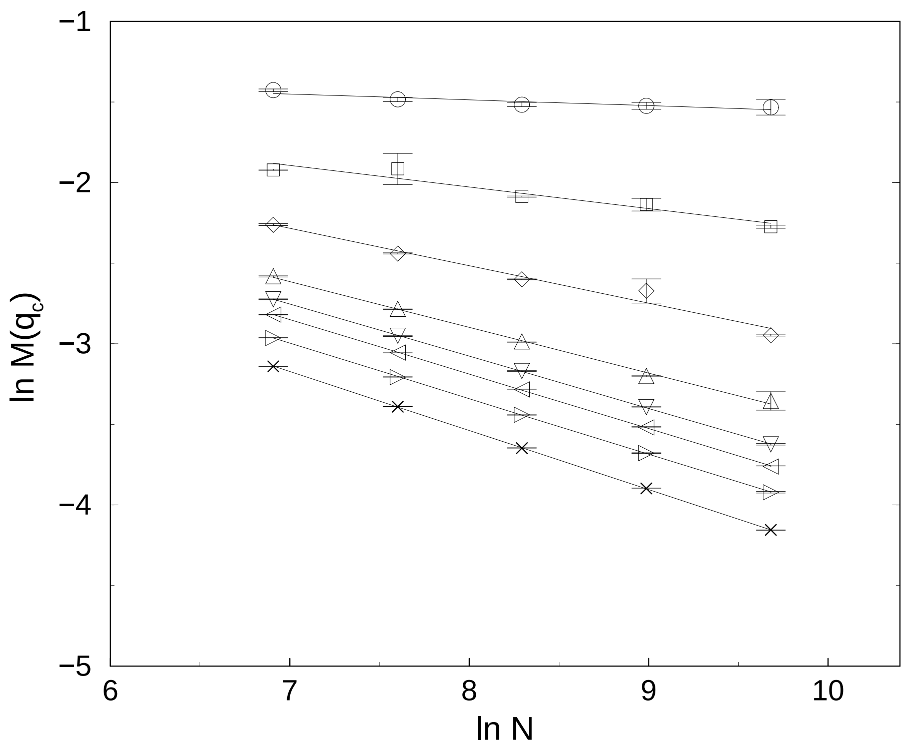

Figure 4: ln $M(q_c)$ versus ln N. From top to bottom, z = 2, 3, 4, 6, 8, 10, 20, and 50.

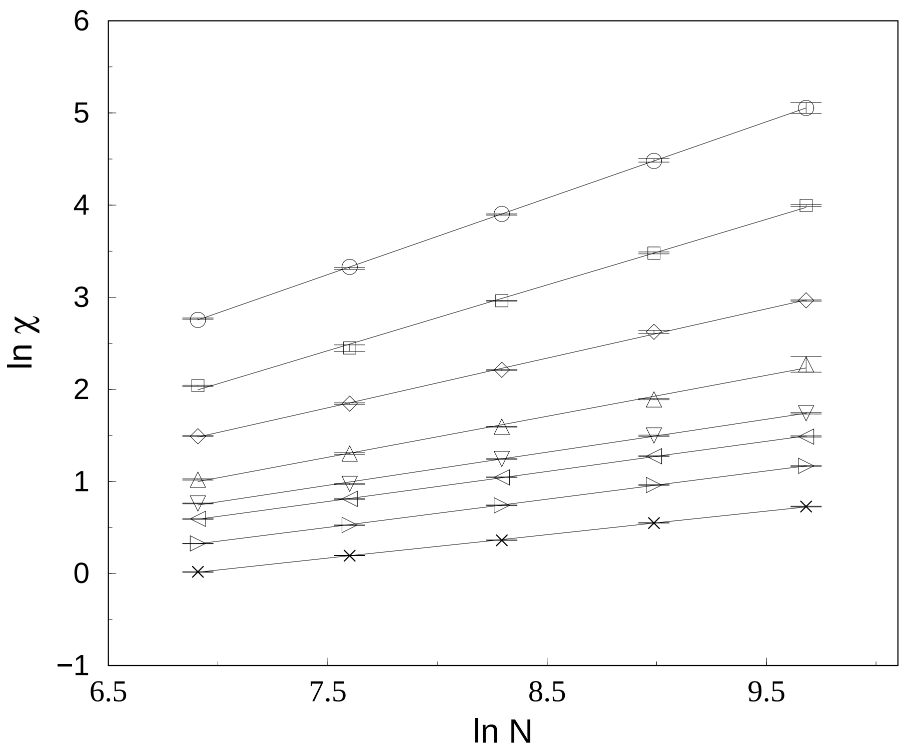

Figure 5: ln χ versus ln N. From top to bottom $z = 2$, 3, 4, 6, 8, 10, 20, and 50.

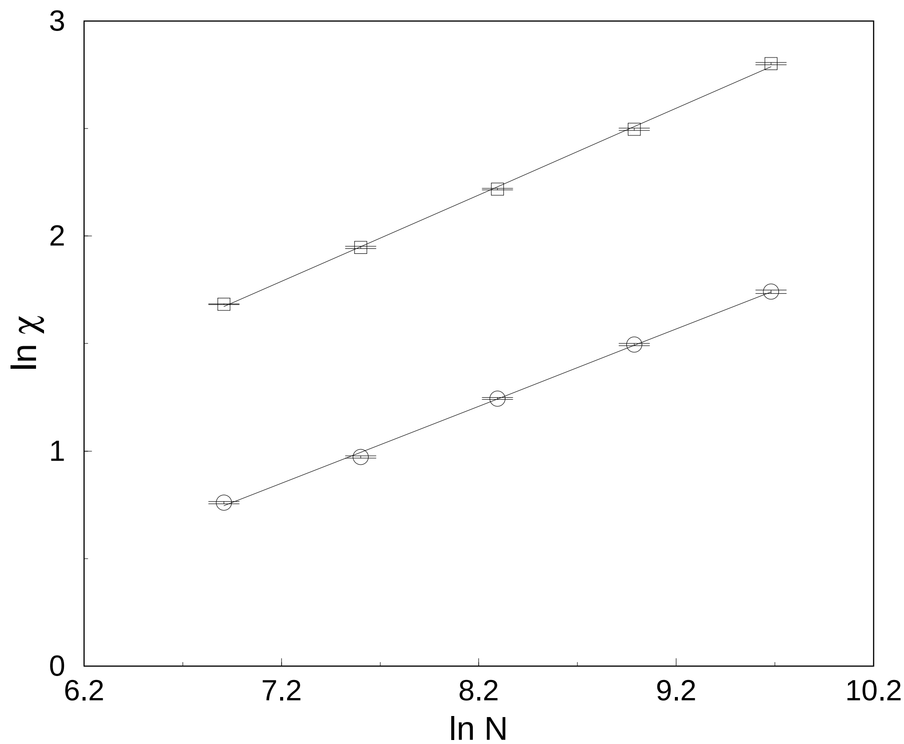

Figure 6: Plot of ln $\chi^{max}(N)$ (square) and ln$\chi(q_c)$ (circle) versus ln N for connectivity $z = 8$.

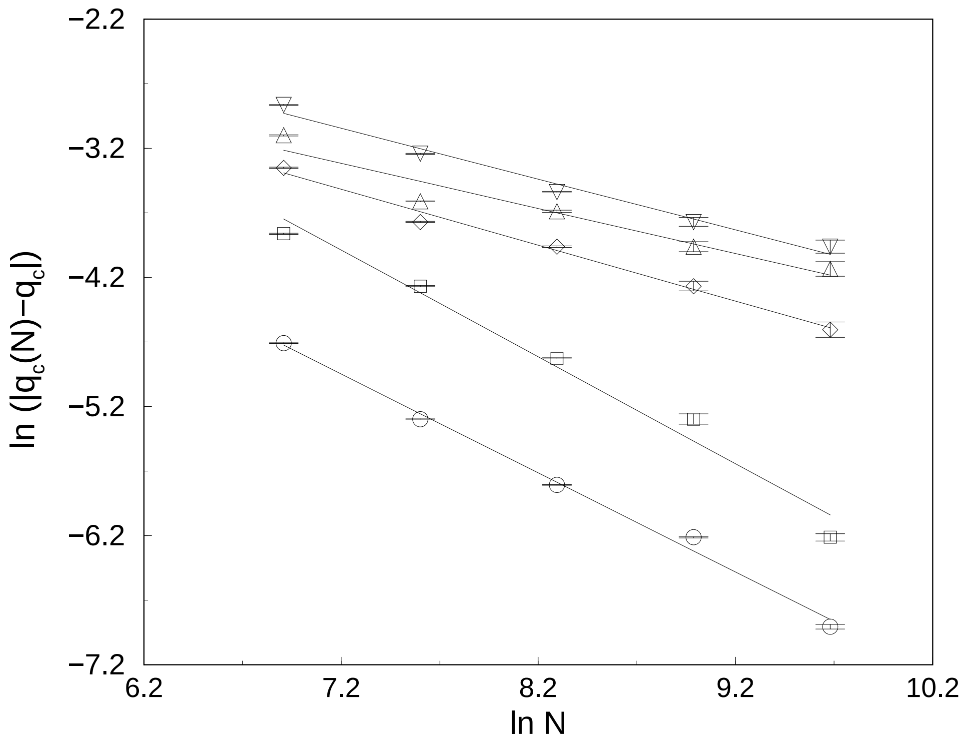

Figure 7: Plot of ln $|q_c(N) - q_c|$ versus ln N. From bottom to top $z = 2$, 3, 4, 6, and 8.

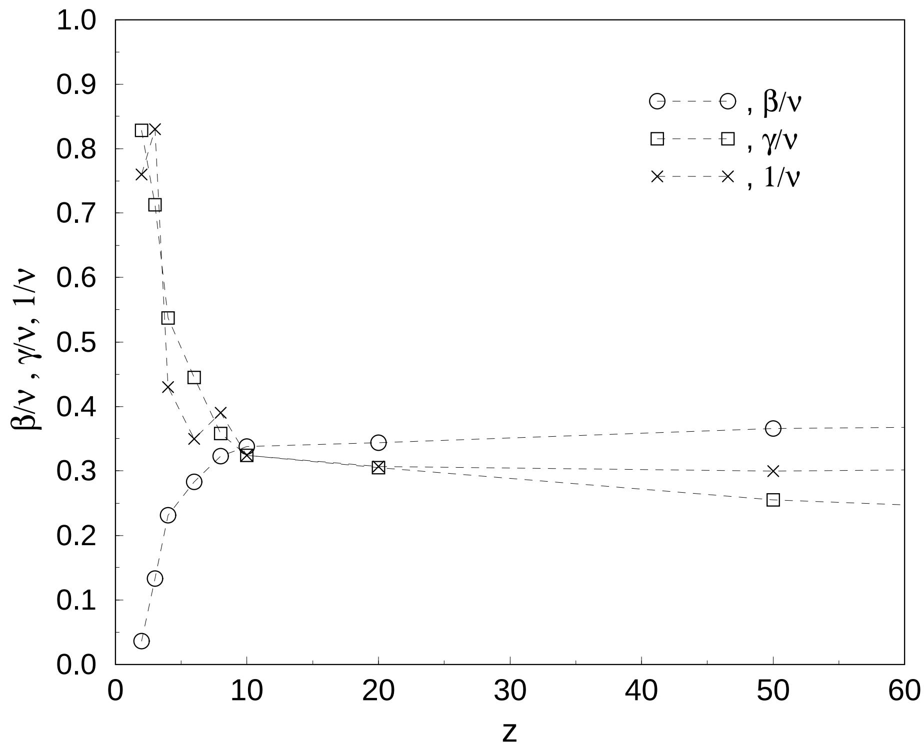

Figure 8: Critical behavior the β/ν, γ/ν, and $1/\nu$ exponents in function of connectivity z.